Extremal problems on the generalized Hua domain of the first type*

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Abstract Some extremal problems between the generalized Hua domain of the first type and the unit ball are studied. The extremal mapping and extremal value in explicit formulas are also obtained.

Keywords: extremal problem, generalized Hua domain of the first type, minimal circumscribed Hermitian ellipsoid.

Let M be a domain in $\boldsymbol{C}^n$ and $p \in M$. Let M_p denote the couple (M, p), a "pointed domain". For two pointed domains M_p and N_q, let Hol (M_p, N_q) denote the set of holomorphic mappings from M to N that send p to q. A mapping $f \in$ Hol (M_p, N_q) is said to be Carathéodory extremal mapping (c-extremal mapping). If

$$|\det \mathrm{d}f(p)| = \sup\{|\det \mathrm{d}g(p)| : g \in \mathrm{Hol}(M_p, N_q)\}, \tag{1}$$

then $|\det \mathrm{d}f(p)|$ is called Carathéodory extremal value (c-extremal value). This is the classical extremal problem. The classical extremal problem is similar to the classical Schwarz lemma,[1] and is considered to be an extension of the classical Schwarz lemma in high dimensions[2]. For the extremal problem, an important part is to determine the explicit formulas for the extremal mapping and extremal value. The c-extremal mapping was first studied by Carathéodory and he obtained the explicit formula for c-extremal mapping from the polydisc into the ball[3]. Explicit formulas for c-extremal mappings and values from the symmetric domains into the ball were obtained by Kubota[4,5]. Ma[2] gave the explicit c-extremal mappings and values from the complex ellipsoid to the unit ball. He also considered a new kind of extremal problem:

Let Q_n denote the set of all couples (M, p), where M is a complex manifold of dimension n and $p \in M$. Let $M_p, N_q \in Q_n$, we say that M_p and N_q are biholomorphically equivalent, and write $M_p \sim N_q$ if there is a map $f \in$ Hol (M_p, N_q) which is a biholomorphism. Obviously, $\sim$ is an equivalence relation. Let $\widetilde{Q}_n = Q_n/\sim$. If $M_p \in Q_n$, let $\widetilde{M}_p$ denote the equivalence class to which M_p belongs. Sometimes we do not distinguish $\widetilde{M}_p$ from M_p if no ambiguity can arise. Define

$$\mu(\widetilde{M}_p, \widetilde{N}_q) = \inf\{-\log |J_{g\circ f}(p)| : f \in \mathrm{Hol}(M_p, N_q), g \in \mathrm{Hol}(N_q, M_p)\}, \tag{2}$$

where M, N are domains in $\boldsymbol{C}^n$, $J_f(p) := \det \mathrm{d}f(p)$ and $\widetilde{M}_p$ denote the equivalence class to which M_p belongs. Ma also obtained the extremal values between the ball and the complex ellipsoid or between the complex ellipsoid and the complex ellipsoid in some cases of this kind of extremal problem. In general, it is very difficult or impossible to obtain explicit formulas for the extremal mapping or value.

In 1998, Yin and Roos constructed four type domains, called super-Cartan domains or Cartan-Hartogs domains[6—8], and the first type super-Cartan domain is

$$Y_I(N; m, n; K) := \{W \in \boldsymbol{C}^N, \boldsymbol{Z} \in \mathfrak{R}_I(m, n) : |W|^{2K} < \det(I - \boldsymbol{Z}\overline{\boldsymbol{Z}}^{\mathrm{T}}), K > 0\},$$

where $\mathfrak{R}_I(m, n)$ denotes the Cartan domain of the first type in the sense of Hua, $\overline{\boldsymbol{Z}}^{\mathrm{T}}$ denotes the conjugate and transposed matrix of $\boldsymbol{Z}$, det denotes the determinant of a square matrix, N are positive integers, and k are positive real numbers. The super-Cartan domains are neither homogeneous domains nor Reinhardt domains.

We have obtained the explicit formulas for the extremal mapping and the value between the ball and the super-Cartan domain of the first type when $k > 1$

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of the above mensioned two extremal problems [9].

Yin[10] constructed four type domains in 2001, called generalized Hua domains, and the first type generalized Hua domain is

$$GHE_I(N_1, \cdots, N_r; m, n; p_1, \cdots, p_r; k) = \left\{ w_j \in \mathbf{C}^{N_j}, Z \in \Re_I(m, n): \sum_{j=1}^{r} | w_j |^{2p_j} < \det(I - Z\overline{Z}^{\mathrm{T}})^k, j = 1, \cdots, r \right\},$$

where $w_j = (w_{j_1}, \cdots, w_j N_j), j = 1, \cdots, r, N_1, \cdots, N_r$ are positive integers and $p_1, \cdots, p_{r-1}, p_r$; k are positive real numbers.

When $r = 1$ and $k = 1$, it is the super-Cartan domain.

In this paper, the explicit formulas for extremal values will be given between the ball and the generalized Hua domain of the first type. These results generalize the results of Refs. [2, 9].

1 Preliminaries

Let T_n denote the subset of Q_n consisting of all pointed taut manifolds, and similarly, $\widetilde{T}_n = T_n / \sim$.

$(\widetilde{T}_n, \mu)$ is a metric space[2].

If both M and N contain 0, we write $JS(M, N) := JS(M_0, N_0)$.

Obviously,

$$\mu(M_0, N_0) = -\log[JS(M, N) \cdot JS(N, M)]. \tag{3}$$

Proposition 1[2]. If D_1, D_2 are belanced domains (i.e. $tz \in D_i$ for $t \in \bar{\triangle}$ and $z \in D_i (i = 1, 2)$, where $\triangle$ denotes the unit disk), and if D_2 is a domain of holomorphy, then any holomorphic map $f \in \mathrm{Hol}((D_1, 0), (D_2, 0))$ satisfies $\mathrm{d}f(0)(D_1) \subset D_2$. Hence,

$$JS(D_1, D_2) = \sup\{ | \det l | : l \text{ complex linear map}, l(D_1) \subset D_2 \}.$$

If both D_1 and D_2 are balanced domains of holomorphy, then

$$\mu((D_1, 0), (D_2, 0)) = \inf\{ -\log | \det(m \cdot l) | : l, m \text{ complex linear maps}, l(D_1) \subset D_2, m(D_2) \subset D_1 \}.$$

In the sequel, we denote $\mu(M, N) := \mu(M_0, N_0)$.

Definition 1[2]. A Hermitian ellipsoid is a domain of the form

$$\left\{ z \in \mathbf{C}^n : \sum_{j,k=1}^{n} a_{jk} z_j \bar{z}_k < 1 \right\},$$

where (a_{jk}) is a positive definite Hermitian matrix.

Proposition 2[2]. Let D be a domain of dimension n containing 0. If l is a complex linear map such that $l(D) \subset B^n$, then $l^{-1}(B^n)$ is a Hermitian ellipsoid containing D. If l is a solution to the extremal problem

$$\sup\{ | \det l | : l \text{complex linear map}, l(D) \subset B^n \},$$

then $l^{-1}(B^n)$ is a circumscribed Hermitian ellipsoid of D of least volume, or minimal circumscribed Hermitian ellipsoid.

Proposition 3[2]. Let D be a bounded domain. Then D has minimal circumscribed Hermitian ellipsoid and the minimal circumscribed Hermitian ellipsoid of D is unique.

For a bounded domain D, let $P(D)$ denote the minimal circumscribed Hermitian ellipsoid. It is easy to check that GHE_I is the balanced domain.

We denote $\mathbf{Z} = (z_{jk})_{m \times n} \in \Re_I(m, n)$ and arrange the elements of the matrix $\mathbf{Z}$ in the form of a vector in $\mathbf{C}^{mn}$:

$$z = (z_{11}, \cdots, z_{1n}, z_{21}, \cdots, z_{2n}, \cdots, z_{mn}).$$

$$\| z \|^2 = \| \mathbf{Z} \|^2 = \mathrm{tr}(\mathbf{Z}\overline{\mathbf{Z}}^{\mathrm{T}}).$$

Sometimes we do not distinguish z from $\mathbf{Z}$ if no ambiguity can arise.

Proposition 4. The minimal circumscribed ellipsoid of the generalized Hua domain of the first type has the form

$$A(a_1, \cdots, a_r, b) = \left\{ (w_1, \cdots, w_r, \mathbf{Z}) \in \mathbf{C}^{\sum_{i=1}^{r} N_i + mn} : \sum_{i=1}^{r} a_i \| w_i \|^2 + b \| \mathbf{Z} \|^2 < 1 \right\}, \tag{4}$$

where $a_i > 0$, $b > 0$ $(i = 1, 2, \cdots, r)$.

Proof. It is similar to the proof of the Proposition 3.1 in Ref. [9].

Proposition 5. When $p_i \geqslant km$ $(i = 1, 2, \cdots, r)$, the unit ball B_{N+mn} is the inscribed unit ball of the generalized Hua domain of the first type, where $N = \sum_{i=1}^{r} N_i$.

Proof. It is similar to the proof of the Proposition 3.2 in Ref. [9].

2 The minimal circumscribed ellipsoid

For any $m\times n$ matrix $\boldsymbol{Z}$ ($m\leqslant n$), there exist unitary matrices $\boldsymbol{U}_{m\times m}$ and $\boldsymbol{V}_{n\times n}$ such that[11]

$$\boldsymbol{Z}=\boldsymbol{U}\begin{pmatrix}\tilde{\lambda}_1 & 0 & \cdots & 0 & 0 & \cdots & 0\\ 0 & \tilde{\lambda}_2 & \cdots & 0 & 0 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & \tilde{\lambda}_m & 0 & \cdots & 0\end{pmatrix}\boldsymbol{V},$$

$$(\tilde{\lambda}_1\geqslant\tilde{\lambda}_2\geqslant\cdots\geqslant\tilde{\lambda}_m\geqslant 0).$$

We denote $\lambda_1=\tilde{\lambda}_1^2,\lambda_2=\tilde{\lambda}_2^2,\cdots,\lambda_m=\tilde{\lambda}_m^2$. Then

$$\|\boldsymbol{Z}\|^2=\tilde{\lambda}_1^2+\tilde{\lambda}_2^2+\cdots+\tilde{\lambda}_m^2=\lambda_1+\lambda_2+\cdots+\lambda_m,$$

$$\det(I-\boldsymbol{Z}\overline{\boldsymbol{Z}}^{\mathrm{T}})=(1-\lambda_1)(1-\lambda_2)\cdots(1-\lambda_m).$$

To obtain the minimal circumscribed ellipsoid on GHE_I, we first find the maximum of the function

$$B'(a_1,\cdots,a_r;b)=\sum_{i=1}^{r}a_ih_i+b\sum_{l=1}^{m}\lambda_l \tag{5}$$

for fixed $a_1,\cdots,a_r,b$ ($a_i>0,b>0(i=1,2,\cdots,r)$) with the constraint

$$G'(h_1,\cdots,h_r;\lambda_1,\cdots,\lambda_m)=\sum_{i=1}^{r}h_i^{p_i}-[(1-\lambda_1)(1-\lambda_2)\cdots(1-\lambda_m)]^k=0, \tag{6}$$

$$(1\geqslant h_j\geqslant 0,j=1,2,\cdots,r;\ 0\leqslant\lambda_l\leqslant 1,l=1,2,\cdots,m),$$

where $h_j=\|w_j\|^2$, $w_j\in\boldsymbol{C}^{N_j}$, $j=1,2,\cdots,r$.

The maximum of the function must exist because the function was discussed on the close set.

By symmetry consideration, the above problem is equal to finding the maximum of the function

$$B(a_1,\cdots,a_r;b)=\sum_{i=1}^{r}a_ih_i+bm\lambda \tag{7}$$

for fixed $a_1,\cdots,a_r,b(a_i>0,b>0(i=1,2,\cdots,r))$ with the constraint

$$G(h_1,\cdots,h_r;\lambda)=\sum_{i=1}^{r}h_i^{p_i}-(1-\lambda)^{km}=0, \tag{8}$$

$$(1\geqslant h_j\geqslant 0,j=1,2,\cdots,r;\ 0\leqslant\lambda\leqslant 1,\lambda_1=\cdots=\lambda_m=\lambda),$$

where $h_j=\|w_j\|^2$, $w_j\in\boldsymbol{C}^{N_j}$, $j=1,2,\cdots,r$.

Now the above conditional extremum problem will be changed to the nonconditional extremum problem:

$$F(h):=F(h_1,\cdots,h_r)=\sum_{i=1}^{r}a_ih_i+bm-bm\Big(\sum_{i=1}^{r}h_i^{p_i}\Big)^{\frac{1}{km}}. \tag{9}$$

If we have obtained the maximum of $F(h)$ and denote it by s, for the fixed $a_1,\cdots,a_r,b(a_i>0,b>0(i=1,2,\cdots,r))$, the circumscribed ellipsoid of the generalized Hua domain of the first type is

$$A(a_1,\cdots,a_r,b)=\Big\{(w_1,\cdots,w_r,\boldsymbol{Z})\in\boldsymbol{C}^{\sum_{i=1}^{r}N_i+mn}:\sum_{i=1}^{r}a_i\|w_i\|^2+b\|\boldsymbol{Z}\|^2<s\Big\}$$

$$=\Big\{(w_1,\cdots,w_r,\boldsymbol{Z})\in\boldsymbol{C}^{\sum_{i=1}^{r}N_i+mn}:\sum_{i=1}^{r}\frac{a_i}{s}\|w_i\|^2+\frac{b}{s}\|\boldsymbol{Z}\|^2<1\Big\}.$$

We then find the minimum of the function

$$\phi(a_1,\cdots,a_r,b)=s^{\sum_{i=1}^{r}N_i+mn}\prod_{i=1}^{r}a_i^{-N_i}b^{-mn}. \tag{10}$$

When we denote by ω_{N+mn} the volume of B_{N+mn}, $\phi(a_1,\cdots,a_r,b)\cdot\omega_{N+mn}$ is the volume of $A(a_1,\cdots,a_r,b)$.

Lemma 1. When $p_i>km\geqslant 1$ ($i=1,2,\cdots,r$), the maximal value of $F(h)$ cannot be attained at a point $(h_1,\cdots,h_r)$ with some $h_i=0$ and some $h_j\neq 0$.

Proof. It is sufficient to consider the point $h^0=(0,h_2^0,\cdots,h_r^0)$ with $D(h):=\sum_{i=1}^{r}h_i^{p_i}=\delta(0<\delta\leqslant 1)$.

Let

$$h^t=(t\delta^{\frac{1}{p_1}},(1-t^{p_1})^{\frac{1}{p_2}}h_2^0,\cdots,(1-t^{p_1})^{\frac{1}{p_r}}h_r^0),\quad t\in(0,1).$$

It is checked that $D(h^t)=\delta$.

One may verify that

$$\lim_{t\to 0^+}\frac{F(h^t)-F(h^0)}{t}=a_1\delta^{\frac{1}{p_1}}>0.$$

Thus, the maximal value cannot be attained at h^0.

Lemma 2. When $p_i>km\geqslant 1(i=1,2,\cdots,r)$, the maximal value of $F(h)$ cannot be attained at a point $(h_1,\cdots,h_r)$ with all $h_i=0(i=1,2,\cdots,r)$.

Proof. Let $h^0=(0,\cdots,0)$, $h^t=(t,0,\cdots,0)$. Then similar to the Proof of Lemma 1, this Lemma can be proved.

By Lemma 1 and 2, we can obtain the following lemma.

Lemma 3. When $p_i > km \geqslant 1$ $(i = 1, 2, \cdots, r)$, the maximal value of $F(h)$ must be attained at a point $(h_1, \cdots, h_r)$ with $0 < h_i < 1$ $(i = 1, 2, \cdots, r)$ and $0 < \sum_{i=1}^{r} h_i^{p_i} \leqslant 1$.

By (9) and $\frac{\partial F(h)}{\partial h_j} = 0$ $(j = 1, 2, \cdots, r)$, we obtain

$$\frac{ka_j}{bp_j} = \left(\sum_{i=1}^{r} h_i^{p_i}\right)^{\frac{1-km}{km}} \cdot h_j^{p_j - 1}, \quad j = 1, 2, \cdots, r. \tag{11}$$

Therefore,

$$\left(\frac{ka_j}{bp_j}\right)^{\frac{p_j}{p_j - 1}} = \left(\sum_{i=1}^{r} h_i^{p_i}\right)^{\frac{1-km}{km} \cdot \frac{p_j}{p_j - 1}} \cdot h_j^{p_j}, \quad j = 1, 2, \cdots, r. \tag{12}$$

Lemma 4. Let $p_i > km \geqslant 1$ $(i = 1, 2, \cdots, r)$ and

$$\left(\frac{ka_1}{bp_1}\right)^{\frac{p_1}{p_1 - 1}} + \left(\frac{ka_2}{bp_2}\right)^{\frac{p_2}{p_2 - 1}} + \cdots + \left(\frac{ka_r}{bp_r}\right)^{\frac{p_r}{p_r - 1}} < 1.$$

Then (11) must have solutions at a point $(h_1, \cdots, h_r)$ with $0 < h_i < 1$ $(i = 1, 2, \cdots, r)$ and $0 < \sum_{i=1}^{r} h_i^{p_i} < 1$, and the solution is unique.

Proof. It is easy to check that

$$p_j > km \geqslant 1 \Longleftrightarrow 0 < \frac{km - 1}{km} \cdot \frac{p_j}{p_j - 1} < 1, \quad j = 1, 2, \cdots, r.$$

We first prove that the solution of (11) is attained at a point $(h_1, \cdots, h_r)$ with $0 < \sum_{i=1}^{r} h_i^{p_i} < 1$ if the solution of (11) exists.

Let

$$E(p_1, \cdots, p_r; m; k) = \max\left\{\frac{(km - 1)p_1}{km(p_1 - 1)}, \frac{(km - 1)p_2}{km(p_2 - 1)}, \cdots, \frac{(km - 1)p_r}{km(p_r - 1)}\right\} (< 1).$$

To seek a contradiction, we suppose that the solution of (11) is attained at a point $(h_1, \cdots, h_r)$ with $\sum_{i=1}^{r} h_i^{p_i} \geqslant 1$. By (12) and the assumption, we know that

$$1 > \left(\frac{ka_1}{bp_1}\right)^{\frac{p_1}{p_1 - 1}} + \left(\frac{ka_2}{bp_2}\right)^{\frac{p_2}{p_2 - 1}} + \cdots + \left(\frac{ka_r}{bp_r}\right)^{\frac{p_r}{p_r - 1}}$$

$$= \left(\sum_{i=1}^{r} h_i^{p_i}\right)^{\frac{(km-1)p_1}{km(p_1 - 1)}} \cdot h_1^{p_1} + \left(\sum_{i=1}^{r} h_i^{p_i}\right)^{\frac{(km-1)p_2}{km(p_2 - 1)}} \cdot h_2^{p_2} + \cdots + \left(\sum_{i=1}^{r} h_i^{p_i}\right)^{\frac{(km-1)p_r}{km(p_r - 1)}} \cdot h_r^{p_r}$$

$$\geqslant \left(\sum_{i=1}^{r} h_i^{p_i}\right)^{-E(p_1, \cdots, p_r; m; k)} \cdots \left(\sum_{i=1}^{r} h_i^{p_i}\right)$$

$$= \left(\sum_{i=1}^{r} h_i^{p_i}\right)^{1 - E(p_1, \cdots, p_r; m; k)} \geqslant 1$$

is not valid.

Next, we prove the solution of (11) must exist and be unique.

Let $H = \sum_{i=1}^{r} h_i^{p_i}$. By (12), we may obtain

$$\sum_{j=1}^{r} \left(\frac{ka_j}{bp_j}\right)^{\frac{p_j}{p_j - 1}} \cdot H^{\frac{(km-1)p_j}{km(p_j - 1)} - 1} = 1. \tag{13}$$

Let

$$f(H) = \sum_{j=1}^{r} \left(\frac{ka_j}{bp_j}\right)^{\frac{p_j}{p_j - 1}} \cdot H^{\frac{(km-1)p_j}{km(p_j - 1)} - 1}.$$

Then

$$\lim_{H \to 1^-} f(H) = \sum_{j=1}^{r} \left(\frac{ka_j}{bp_j}\right)^{\frac{p_j}{p_j - 1}} < 1,$$

$$\lim_{H \to 0^+} f(H) = +\infty,$$

and

$$f'(H) = \sum_{j=1}^{r} \left(\frac{ka_j}{bp_j}\right)^{\frac{p_j}{p_j - 1}} \cdot \frac{-(p_j - km)}{km(p_j - 1)} \cdot H^{\frac{(km-1)p_j}{km(p_j - 1)} - 2} < 0.$$

Thus, $f(H)$ is a decreasing function strictly in $(0, 1)$.

Therefore, (13) has a unique solution in $(0, 1)$. Then we may obtain the unique solution from (12) under determined H.

Lemma 5. Under the conditions of Lemma 4, the maximal value of $F(h)$ is reached at its unique steady point.

Proof. By direct computations, we can verify that $F(h)$ reaches the maximal value at its unique steady point.

Lemma 6. Let $p_i > km \geqslant 1$ $(i = 1, 2, \cdots, r)$ and

$$\left(\frac{ka_1}{bp_1}\right)^{\frac{p_1}{p_1 - 1}} + \left(\frac{ka_2}{bp_2}\right)^{\frac{p_2}{p_2 - 1}} + \cdots + \left(\frac{ka_r}{bp_r}\right)^{\frac{p_r}{p_r - 1}} \geqslant 1.$$

Then (11) must have no solution at a point $(h_1, \cdots,$

h_r) with $0 < h_i < 1$ ($i = 1, 2, \cdots, r$) and $0 < \sum_{i=1}^{r} h_i^{p_i} < 1$.

Proof. Seeking a contradiction, suppose that the solution of (11) is attained at a point ($h_1, \cdots, h_r$) with $0 < \sum_{i=1}^{r} h_i^{p_i} < 1$.

When $km \neq 1$,

$$1 \leqslant \left(\frac{ka_1}{bp_1}\right)^{\frac{p_1}{p_1-1}} + \left(\frac{ka_2}{bp_2}\right)^{\frac{p_2}{p_2-1}} + \cdots + \left(\frac{ka_r}{bp_r}\right)^{\frac{p_r}{p_r-1}}$$
$$= \left(\sum_{i=1}^{r} h_i^{p_i}\right)^{\frac{(km-1)p_1}{km(p_1-1)}} \cdot h_1^{p_1} + \left(\sum_{i=1}^{r} h_i^{p_i}\right)^{\frac{(km-1)p_2}{km(p_2-1)}} \cdot h_2^{p_2}$$
$$+ \cdots + \left(\sum_{i=1}^{r} h_i^{p_i}\right)^{\frac{(km-1)p_r}{km(p_r-1)}} \cdot h_r^{p_r}$$
$$\leqslant \left(\sum_{i=1}^{r} h_i^{p_i}\right)^{-E(m, p_1, \cdots, p_r)} \cdots \left(\sum_{i=1}^{r} h_i^{p_i}\right)$$
$$= \left(\sum_{i=1}^{r} h_i^{p_i}\right)^{1-E(m, p_1, \cdots, p_r)} < 1$$

is not valid.

When $km = 1$, (11) becomes $\frac{ka_j}{bp_j} = h_j^{p_j - 1}$ ($j = 1, 2, \cdots, r$). Thus,

$$1 \leqslant \sum_{i=1}^{r} \left(\frac{ka_i}{bp_i}\right)^{\frac{p_i}{p_i-1}} = \sum_{i=1}^{r} h_i^{p_i} < 1$$

is not valid.

Lemma 7. Under the constraint $\sum_{i=1}^{r} h_i^{p_i} = 1$, $F(h)$ must have the maximal value at the point ($h_1, \cdots, h_r$) with $0 < h_i < 1$ ($i = 1, 2, \cdots, r$), and the maximal value is unique.

Proof. We first prove that $F(h)$ has the maximal value if the conditional extremum on $F(h)$ exists. Let

$$J(h_1, \cdots, h_r, \lambda) = \sum_{i=1}^{r} a_i h_i + \lambda\left(1 - \sum_{i=1}^{r} h_i^{p_i}\right).$$

Suppose

$$\frac{\partial J(h_1, \cdots, h_r, \lambda)}{\partial h_j} = a_j - \lambda p_j h_j^{p_j - 1} = 0, \quad j = 1, 2, \cdots, r.$$

Thus

$$a_j = \lambda p_j h_j^{p_j - 1}, \quad j = 1, 2, \cdots, r. \tag{14}$$

Furthermore, it is easy to check that $F(h)$ has the maximal value if the conditional extremum on $F(h)$ exists.

We then prove that the conditional extremum on $F(h)$ exists and is unique. By (14), we have

$$h_j^{p_j} = \left(\frac{a_j}{\lambda p_j}\right)^{\frac{p_j}{p_j-1}}, \quad (j = 1, 2, \cdots, r).$$

Thus,

$$\sum_{i=1}^{r} \left(\frac{a_i}{\lambda p_i}\right)^{\frac{p_i}{p_i-1}} = 1. \tag{15}$$

Denoting $L(\lambda) = \sum_{i=1}^{r} \left(\frac{a_i}{\lambda p_i}\right)^{\frac{p_i}{p_i-1}}$, then

$$L'(\lambda) = -\sum_{i=1}^{r} \left(\frac{p_i}{p_i - 1}\right)\left(\frac{a_i}{p_i}\right)^{\frac{p_i}{p_i-1}} \cdot \lambda^{-1} < 0.$$

Therefore, $L(\lambda)$ is a decreasing function strictly on λ. However,

$L(\lambda) \to +\infty$, when $\lambda \to 0+$;
$L(\lambda) \to 0$, when $\lambda \to +\infty$.

Thus, (15) uniquely determines λ, and (14) uniquely determines h_j, $j = 1, 2, \cdots, r$.

Now we can compute the minimal circumscribed ellipsoid of the generalized Hua domain of the first type.

When $p_j > km \geqslant 1$ ($j = 1, 2, \cdots, r$), and

$$\left(\frac{ka_1}{bp_1}\right)^{\frac{p_1}{p_1-1}} + \left(\frac{ka_2}{bp_2}\right)^{\frac{p_2}{p_2-1}} + \cdots + \left(\frac{ka_r}{bp_r}\right)^{\frac{p_r}{p_r-1}} < 1. \tag{16}$$

By calculations, the extreme value problem above has a unique solution (and the stable point is unique). The extreme point is

$$\frac{a_j}{s} = \frac{N_j \cdot \left(n + k\sum_{i=1}^{r}\left(\frac{N_i}{p_i}\right)\right)^{\frac{km}{p_j}}}{\left(\sum_{i=1}^{r} N_i + mn\right) \cdot \left(\frac{N_j}{p_j}\right)^{\frac{1}{p_j}} \cdot \left(\sum_{i=1}^{r}\left(\frac{N_i}{p_i}\right)\right)^{\frac{km-1}{p_j}} \cdot k^{\frac{km}{p_j}}}, \quad j = 1, 2, \cdots, r; \tag{17}$$

$$\frac{b}{s} = \frac{n + k\sum_{i=1}^{r}\left(\frac{N_i}{p_i}\right)}{\left(\sum_{i=1}^{r} N_i + mn\right)}. \tag{18}$$

We can also verify that the function ϕ really reaches the minimal value.

When $p_j > km \geqslant 1$ $(j = 1, 2, \cdots, r)$, and

$$\left(\frac{ka_1}{bp_1}\right)^{\frac{p_1}{p_1-1}} + \left(\frac{ka_2}{bp_2}\right)^{\frac{p_2}{p_2-1}} + \cdots + \left(\frac{ka_r}{bp_r}\right)^{\frac{p_r}{p_r-1}} \geqslant 1. \tag{19}$$

It is easy to know that the function ϕ must reach the minimum on the curve

$$\left(\frac{ka_1}{bp_1}\right)^{\frac{p_1}{p_1-1}} + \left(\frac{ka_2}{bp_2}\right)^{\frac{p_2}{p_2-1}} + \cdots + \left(\frac{ka_r}{bp_r}\right)^{\frac{p_r}{p_r-1}} = 1 \tag{20}$$

under condition (19). Curve (20) is the boundary of domain (16).

From the discussion above and noting that the stable point is unique, we obtain that the function ϕ must reach the minimal value at the extreme point (17), (18).

Therefore, we obtain the minimal circumscribed ellipsoid of the generalized Hua domain of the first type

$$A_{\min} = \Big\{ (w_1, \cdots, w_r, Z) \in C^{\sum_{i=1}^{r} N_i + mn} : \sum_{j=1}^{r} c_j \| w_j \|^2 + d \| Z \|^2 < 1 \Big\}, \tag{21}$$

where

$$c_j = \frac{N_j \cdot \left(n + k \sum_{i=1}^{r} \left(\frac{N_i}{p_i}\right) \right)^{\frac{km}{p_j}}}{\left(\sum_{i=1}^{r} N_i + mn \right) \cdot \left(\frac{N_j}{p_j}\right)^{\frac{1}{p_j}} \cdot \left(\sum_{i=1}^{r} \left(\frac{N_i}{p_i}\right) \right)^{\frac{km-1}{p_j}} \cdot k^{\frac{km}{p_j}}},$$

$$j = 1, 2, \cdots, r;$$

$$d = \frac{n + k \sum_{i=1}^{r} \left(\frac{N_i}{p_i}\right)}{\left(\sum_{i=1}^{r} N_i + mn \right)}.$$

3 The extremal mapping

Now we can obtain the following results:

Theorem 1. When $p_i > km \geqslant 1$, $i = 1, \cdots, r$, an extremal mapping from the generalized Hua domain of the first type

$GHE_I(N_1, \cdots, N_r;\ m, n; p_1, \cdots, p_r; k)$ to the ball $B_{\sum_{i=1}^{r} N_i + mn}$ is

$$f: GHE_I(N_1, \cdots, N_r; m, n; p_1, \cdots, p_r; k) \to B_{\sum_{i=1}^{r} N_i + mn}.$$

$$f_{jl}((w_1, \cdots, w_r, Z)) = a_j^{\frac{1}{2}} w_{jl},$$

$$j = 1, 2, \cdots, r, \quad l = 1, 2, \cdots, N_j.$$

$$f_{uv}((w_1, \cdots, w_r, Z)) = b^{\frac{1}{2}} z_{uv},$$

$$u = 1, 2, \cdots, m, \quad v = 1, 2, \cdots, n.$$

Here, $w_j = (w_{j1}, \cdots, w_{jN_j})$, $j = 1, \cdots, r$, and

$$a_j = \frac{N_j \cdot \left(n + k \sum_{i=1}^{r} \left(\frac{N_i}{p_i}\right) \right)^{\frac{km}{p_j}}}{\left(\sum_{i=1}^{r} N_i + mn \right) \cdot \left(\frac{N_j}{p_j}\right)^{\frac{1}{p_j}} \cdot \left(\sum_{i=1}^{r} \left(\frac{N_i}{p_i}\right) \right)^{\frac{km-1}{p_j}} \cdot k^{\frac{km}{p_j}}},$$

$$j = 1, \cdots, r;$$

$$b = \frac{n + k \sum_{i=1}^{r} \left(\frac{N_i}{p_i}\right)}{\sum_{i=1}^{r} N_i + mn}.$$

Proof. According to Proposition 1, Proposition 2, Definition 1 and (21), we know that this is true.

Theorem 2. When $p_i > km \geqslant 1$, $i = 1, \cdots, r$, the extremal value between the generalized Hua domain of the first type $GHE_I(N_1, \cdots, N_r; m, n; p_1, \cdots, p_r; k)$ and the ball $B_{\sum_{i=1}^{r} N_i + mn}$ is

$$JS(GHE_I(N_1, \cdots, N_r; m, n; p_1, \cdots, p_r; k), B_{\sum_{i=1}^{r} N_i + mn})$$

$$= \left\{ \frac{\prod_{i=1}^{r} N_i^{N_i} \cdot \left(n + k \sum_{i=1}^{r} \left(\frac{N_i}{p_i}\right) \right)^{m\left(n + k \sum_{i=1}^{r} \left(\frac{N_i}{p_i}\right)\right)}}{(N + mn)^{N+mn} \prod_{i=1}^{r} \left(\frac{N_i}{p_i}\right)^{\left(\frac{N_i}{p_i}\right)} \cdot \left(\sum_{i=1}^{r} \left(\frac{N_i}{p_i}\right) \right)^{(km-1)\left(\sum_{i=1}^{r}\left(\frac{N_i}{p_i}\right)\right)} \cdot k^{km \sum_{i=1}^{r} \left(\frac{N_i}{p_i}\right)}} \right\}^{\frac{1}{2}},$$

$$\mu(GHE_I(N_1, \cdots, N_r; m, n; p_1, \cdots, p_r; k), B_{\sum_{i=1}^{r} N_i + mn})$$

$$= \frac{1}{2}\log \frac{(N+mn)^{N+mn}\prod_{i=1}^{r}\left(\frac{N_i}{p_i}\right)^{\left(\frac{N_i}{p_i}\right)}\cdot\left(\sum_{i=1}^{r}\left(\frac{N_i}{p_i}\right)\right)^{(km-1)\left(\sum_{i=1}^{r}\left(\frac{N_i}{p_i}\right)\right)}\cdot k^{km\sum_{i=1}^{r}\left(\frac{N_i}{p_i}\right)}}{\prod_{i=1}^{r}N_i^{N_i}\left(n+k\sum_{i=1}^{r}\left(\frac{N_i}{p_i}\right)\right)^{m\left(n+k\sum_{i=1}^{r}\left(\frac{N_i}{p_i}\right)\right)}},$$

where $N = \sum_{i=1}^{r} N_i$.

Proof. According to the definition of $JS(M_0, N_0)$ and Theorem 1, by direct computation, we can obtain this result.

Remark. When $m = 1$ and $k = 1$, the generalized Hua domain of the first type is a complex ellipsoid. Our results are the same as the results in Ref. [2].

When $r = 1$ and $k = 1$, the generalized Hua domain of the first type is the super-Cartan domain of the first type. Our results are the same as the results in Ref. [9].

References

1 Travaglini G. An analogue of the Schwarz lemma for bounded symmetric domains. Proc. Amer. Math. Soc., 1983, 83: 85—88.

2 Ma D. W. Caratheodory extremal maps of ellipsoids. J. Math. Soc. Japan, 1997, 49(4): 723—739.

3 Caratheodory C. Uber die abbildungen, die durch systeme von analytischen funktionen von mehreren veranderlichen erzeugt werden. Math. Z., 1932, 34: 758—792.

4 Kubota Y. An extremal problem on bounded symmetric domains. Bull. London Math. Soc., 1983, 15: 126—130.

5 Kubota Y. An extremal problem on the classical Cartan domains Ⅲ. Kodai Math. J., 1982, 5(2): 402—407.

6 Yin W. P. The Bergman kernel function on super-Cartan domains of the first type. Science in China (Series A), 2000, 43(1): 13—21.

7 Yin W. P. The Bergman kernel function on super-Cartan domains of the fouth type. Acta Math. Sinica (in Chinese), 1999, 42(5): 951—960.

8 Yin W. P. The Bergman kernels on Cartan-Hartogs domains. Chinese Science Bulletin, 1999, 44(21): 1947—1951.

9 Yin W. P. and Su J. B. Extremal problems on super-Cartan domain of the first type. Complex Variables, 2003, 48(5): 441—452.

10 Yin W. P. and Su J. B. Bergman kernels on generalized Hua domains. Progress in Natural Science, 2002, 12(12): 893—899.

11 Lu Q. K. The Classical Manifolds and the Classical Domains. Shanghai: Shanghai Scientific & Technical Publishers (in Chinese), 1963.